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ADDENDUM

Phase-induced atomic permutations in icosahedral quasicrystals are related to stellated polyhedra

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Abstract. By certain variations of the phase of quasiperiodic tilings, vertices can be permuted and even transported to infinity in a diffusive way. In a preceding article, we have studied such a motion for the icosahedral Ammann–Kramer–Penrose tiling. The 'quantum of diffusion' occurs in a triacontahedral cage, where atoms in sets of three and seven can be interchanged arbitrarily. Here we show that these kinetic processes can be represented by the 'stellated polyhedra' $\{5, \frac{5}{2}\}$ and $\{3, \frac{5}{2}\}$, respectively.

When the phase of a quasiperiodic tiling is changed, vertices are flipping and cause a rearrangement of the tiles. For the octagonal Ammann–Beenker tiling [1] Frenkel *et al* [2] have demonstrated that the vertices ('atoms') do not only move back and forth. Instead, there are octagonal cages, containing a small octagon with three of its vertices occupied by atoms (figure 1(*a*)). In a succession of eight flips these atoms jump from vertex to vertex of this octagon, until they are permuted cyclically. It has been shown that by a sequence of permutations in adjacent cages atoms can be transported along larger and larger rings and even to infinity [3, 4].

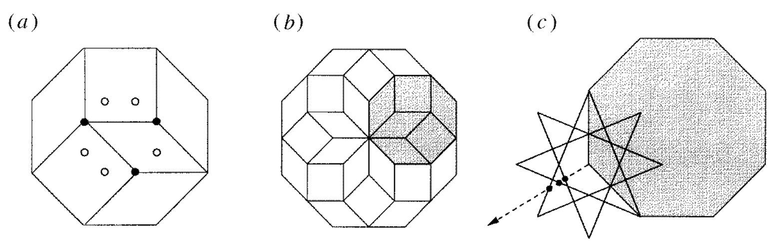

Figure 1. (*a*) An octagonal cage with eight optional vertex positions, three of which are occupied. (*b*) Eight atomic surfaces corresponding to the eight vertex positions. (*c*) Each atomic surface is represented as a straight line; the eight lines form a stellated octagon $\{\frac{8}{3}\}$. Every ray intersects the polygon in three points, which are permuted when the ray rotates once around the central point.

Recently, we have investigated the elementary permutation process in the three-dimensional icosahedral Ammann–Kramer–Beenker tiling [5]. There, the role of the octagonal cage is played by a rhombic triacontahedron. Instead of the inner octagon, there are two intertwined polyhedra, an icosahedron of twelve vertices, occupied by three atoms, and a dodecahedron of 20 vertices, occupied by seven atoms. Through successive flips, corresponding to well defined changes of the phase, the three atoms can be transported on the icosahedron and permuted, and the same can happen with the seven atoms on the dodecahedron.

In this addendum, we are going to prove that the kinetic properties of the phason induced diffusion can be represented in an appealing geometric representation by 'stellated' polygons and polyhedra.

We first describe the picture for the octagonal case. The eight optional vertex positions in the interior of the octagonal cage correspond to eight octagonal atomic surfaces, whose projections into perpendicular space E' form eight octagons sharing one vertex (figure 1(b)). On a circle about this *central* vertex each octagon takes a sector of 135°. As long as the phase is contained in this sector, the corresponding vertex in physical space E is occupied by an atom. Three octagons overlap in a sector of 45°, hence each such sector describes a configuration of the three atoms inside the cage. We now represent each octagonal surface in E' by a straight line, connecting the outmost vertices of the octagonal atomic surface in the 135° sector and perpendicular to the line between central vertex and midpoint of the octagon.

The eight lines intersect and form a stellated regular octagon (figure 1(c)). It is denoted $\{\frac{8}{3}\}$, as each of its edges covers an angle of $\frac{2\pi}{8/3}$. It has the 'density' three, because a ray emanating from the central vertex intersects three of its edges, and it depicts a three-sheeted covering of the circle with the central vertex as branch point. If the ray sweeps over one of the vertices of the stellated octagon, the intersection point leaves one edge and enters another one, which corresponds to a flip inside the octagonal cage. If one follows the intersection point, then after a full turn one arrives at another edge corresponding to the permuted atom. Thus the 'quantum of diffusion' of the Ammann–Beenker tiling is well described by the stellated octagon.

The picture is easily transferred to the icosahedral case. The 32 optional vertex positions inside the triacontahedral cage correspond to 32 triacontahedral atomic surfaces, whose projections into the three-dimensional perpendicular space also form 32 triacontahedra sharing a central vertex. For twelve of them, the central vertex is a five-fold one, for 20 a three-fold one. Now we take the twelve triacontahedra and represent each by a pentagon, perpendicular to the line connecting its midpoint with the central vertex. The resulting polyhedron consists of 12 interwoven pentagons (figure 2(a)). They intersect near each vertex in the same manner as the edges of a stellated pentagon $\{\frac{5}{2}\}$, known as 'pentagram' [6]. Hence the polyhedron is denoted with the Schläfli symbol $\{5, \frac{5}{2}\}$ [7]. It has a density of three. Thus a ray emanating from the central vertex intersects three of the faces. These correspond to the occupied vertices inside the triacontahedral cage and on the small icosahedron. In total the polyhedron is a stellated dodecahedron of Euler characteristic minus six and with 12 prongs. The tips of the 12 prongs are branching points, i.e. if they are encircled by the ray, two of the three occupied positions are exchanged. By encircling several prongs one can obtain any permutation of the three atoms.

The 20 remaining triacontahedra are analogously represented by large triangles, which intersect each other forming the stellated icosahedron $\{3, \frac{5}{2}\}$ (figure 2(b)). It has density seven and also 12 branch points, but Euler characteristic two, and it represents the kinetic situation of the remaining seven atoms in the triacontahedral cage.

If intersections of faces are permitted, the stellated dodecahedron and icosahedron together with their dual polyhedra $\{\frac{5}{2}, 5\}$ and $\{\frac{5}{2}, 3\}$ form the only regular polyhedra in three dimensions, apart from the five Euclidean ones.

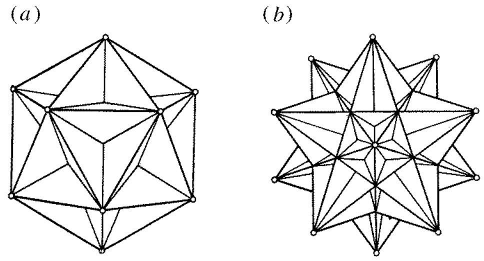

Figure 2. (*a*) The polyhedra $\{5, \frac{5}{2}\}$ formed by 12 intersecting pentagons. (*b*) The polyhedra $\{3, \frac{5}{2}\}$ formed by 20 intersecting triangles. From [7].

Thus, by a dimensional reduction of the atomic surfaces we have arrived at a visualization of the coverings representing the elementary diffusion process of quasiperiodic tilings. The permutations of vertices within a triacontahedral cage in the Ammann–Kramer–Penrose tiling are fully determined by the topological structure of these generalized polyhedra. The dimensional reduction is also possible for many other tilings fulfilling the closeness condition [8], which means that an extended quasiperiodic atomic hyperplane can be constructed. The method could, for example, be applied to determine the phase-induced permutations in the four-dimensional quasicrystal derived from the E_8 lattice [9].

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